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Scaling Inequalities for Oriented Percolation

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We look at seven critical exponents associated with two-dimensional oriented percolation. Scaling theory implies that these quantities satisfy four equalities. We prove five related inequalitites.

KEY WORDS: Critical phenomena; scaling relations; critical exponents; correlation lengths; renormalized bond construction.

1. INTRODUCTION

We begin by describing the model. Although it can be defined for any dimension, we will restrict ourselves to the two-dimensional case. Let $\mathscr{L} = \{(m, n) \in \mathbb{Z}^2;\ m+n \text{ is even, } n \geqslant 0\}$. From each $z \in \mathscr{L}$ there is an oriented arc to $z+(1, 1)$ and to $z+(-1, 1)$. Each arc, also called a bond, is independently open with probability p and closed with probability $1-p$. We think of an open bond as allowing us to along it in the direction of orientation. With this in mind we define the following:

$x \to y$ (y can be reached from x) if there is an open path from x to y, that is, there is a sequence $x = x_0, x_1,..., x_m = y$ of points in $\mathscr{L}$ such that for each $k \leqslant m$ the arc from x_{k-1} to x_k is open.

$$C_0 = (\text{the cluster containing the origin } (0, 0)) = \{x: 0 \to x\}$$
$$\Omega_\infty = \{|C_0| = \infty\} = \text{“percolation occurs”}$$

Here $|A|$ denotes the cardinality of A.

The event Ω_∞ has zero probability when p is small and positive probability when p is close to 1. As the value of p increases, the system undergoes a “phase transition” at $p_c = \inf\{p:\ P_p(\Omega_\infty) > 0\}$.[2] Here, we

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study the critical exponents associated with the phase transition. To define these quantities, we start with β, the exponent associated with the percolation probability. Intuitively β measures the rate at which $P_p(\Omega_\infty)$ decreases to zero as p approaches p_c. We expect that

$$P_p(\Omega_\infty) \sim C(p-p_c)^\beta$$

where $\sim$ means

$$\lim_{p \downarrow p_c} \frac{P_p(\Omega_\infty)}{C(p-p_c)^\beta} = 1$$

But following common practice, we use the weaker definition

$$P_p(\Omega_\infty) \approx (p-p_c)^\beta$$

where $\approx$ means

$$\lim_{p \downarrow p_c} \frac{\log P_p(\Omega_\infty)}{\beta \log(p-p_c)} = 1$$

The second critical exponent γ cornerns the means cluster size $E_p|C_0|$:

$$E_p|C_0| \approx (p_c - p)^{-\gamma} \qquad \text{as} \quad p \uparrow p_c$$

To extend the definition to the supercritical case, we restrict to the event that the cluster is finite and then define γ' by

$$E_p\{|C_0|, |C_0| < \infty\} \approx (p-p_c)^{-\gamma'} \qquad \text{as} \quad p \downarrow p_c$$

The definitions above are analogous to the ones in the theory of ordinary (unorieted) percolation. The next quantity has no analogue in that theory. Let

$$\bar{\xi}_n = \{x: \text{ there is a } y \leqslant 0 \text{ so that } (y, 0) \to (x, n)\}$$

In words, $\bar{\xi}_n$ is the state at time n starting from $\bar{\xi}_0 = \{0, -2, -4, \ldots\}$. Let

$$\bar{r}_n = \sup \bar{\xi}_n$$

It is known (see ref. 2, pp. 1005–1006) that

$$\frac{\bar{r}_n}{n} \to \alpha(p) \qquad \text{almost surely} \quad \text{as } n \to \infty$$

and that

$$p_c = \inf\{p: \alpha(p) > 0\}$$

We define the critical exponent σ associated with the "edge speed" $\alpha(p)$ by

$$\alpha(p) \approx (p - p_c)^\sigma \qquad \text{as} \quad p \downarrow p_c$$

The quantities we have defined so far concern the behavior of the system as p approaches the critical probability p_c. The next two concern the behavior at the critical value p_c. Let

$$\xi_n^0 = \{x\colon (0, 0) \to (x, n)\}$$

We define the critical exponent for the survival probability by

$$P_{\text{cr}}(\xi_n^0 \neq \varnothing) \approx n^{-1/\delta_r}$$

where the subscript cr indicates we are considering $p = p_c$. The r here is for radius (of the cluster) and is included to make our definition match the one for ordinary percolation.[9]

The second quantity at criticality is related to the mean cluster size as a function of the time n:

$$E_{\text{cr}} |\xi_n^0| \approx n^\eta, \qquad 0 \leqslant \eta \leqslant 1$$

While the definition of δ_r is analogous to the one for ordinary percolation, the definition of η is different from its counterpart:

$$P_{\text{cr}}(0 \to x) \approx |x|^{2-d-\eta'} \qquad \text{as} \quad |x| \to \infty$$

where $|x| = \max(|x_1|, |x_2|)$ if $x = (x_1, x_2)$. To relate the two definitions, observe that

$$\sum_{x\colon |x| = n} P_{\text{cr}}(0 \to x) \approx n^{1-\eta'}$$

Hence our η is like $1 - \eta'$.

Last but not least we come to the correlation lengths. We use the definitions introduced and explained in the companion paper.[5] If we let $\tau^A = \inf\{n\colon \xi_n^A = \varnothing\}$ and write τ^0 when $A = \{0\}$, then the parallel correlation length $L_{||}(p)$ can be defined in the subcritical case by

$$[L_{||}(p)]^{-1} = \lim_{n \to \infty} [-(1/n) \log P_p(\tau^0 > n)]$$

The associated critical exponent $\nu_{||}$ is defined by

$$L_{||}(p) \approx (p_c - p)^{-\nu_{||}}$$

Let r_n^0 denote the rightmost site in ξ_n^0, i.e.,

$$r_n^0 = \sup \xi_n^0 \qquad (= -\infty \quad \text{if } \xi_n^0 = \varnothing)$$

and let

$$R^0 = \sup_n r_n^0$$

The perpendicular correlation length $L_\perp(p)$ for the subcritical case is defined by

$$[L_\perp(p)]^{-1} = \lim_{n\to\infty} [(-1/n) \log P_p(R^0 > n)]$$

and $\nu_\perp$ by

$$L_\perp(p) \approx (P_c - p)^{-\nu_\perp}$$

For the supercritical case there are also two correlation lengths. First, the parallel one. $L_{||}(p)$ is defined by

$$[L_{||}(p)]^{-1} = \lim_{n\to\infty} [(-1/n) \log P_p(n < \tau^0 < \infty)]$$

and $\nu'_{||}$ by

$$L_{||}(p) \approx (p - p_c)^{-\nu'_{||}}$$

(The prime on $\nu_{||}$ is to indicate that we are now looking at the limit as $p \downarrow p_c$.)

Extrapolating from the first three definitions, the reader might expect the last one to be

$$[L_\perp(p)]^{-1} = \lim_{n\to\infty} [(-1/n) \log P_p(R^0 > n, \tau^0 < \infty)]$$

For the results we will prove below it is convenient to use

$$[L_\perp^\tau(p)]^{-1} = \lim_{n\to\infty} [(-1/n) \log P_p(\tau^{(-2n,0)} < \infty)]$$

instead. This is supported by (1.9) in ref. 5, which, together with Lemma 3 in ref. 4, gives us $L_\perp(p) \leqslant L_\perp^\tau(p) \leqslant 2L_\perp(p)$. The associated critical exponent $\nu'_\perp$ is defined by

$$L_\perp^\tau(p) \approx (p - p_c)^{-\nu'_\perp}$$

Having introduced the critical exponents, we turn now to the results. Scaling theory predicts that

$$\gamma = \nu'_{||} + \nu'_{\perp} - 2\beta \tag{1.1}$$

$$\sigma = \nu'_{||} - \nu'_{\perp} \tag{1.2}$$

$$\beta = \nu'_{||}/\delta_r \tag{1.3}$$

$$\gamma = (\eta + 1)\nu_{||} \tag{1.4}$$

The first three equalities can be found in ref. 7, while the last one is in ref. 1. Those papers use the notation of Reggeon field theory, so to get the results above one has to change variables:

$$\delta_r = 1/\delta, \qquad \nu_{||} = \nu, \qquad \nu_{\perp} = (z/2)\nu, \qquad \alpha = v$$

The main purpose of this work is to prove some inequalities related to (1.1)–(1.4). In Section 2 we show that

$$E_p|C_0| \leqslant 10 L_{||}(p)\, L_{\perp}(p) \qquad \text{when} \quad L_{||}(p), L_{\perp}(p) \geqslant 1 \tag{1.5}$$

or in terms of critical exponents,

$$\gamma \leqslant \nu_{||} + \nu_{\perp} \tag{1.6}$$

Comparing this with (1.1) shows that -2β is missing from the right-hand side.

In Section 3 we show that

$$L^{\tau}_{\perp}(p) \leqslant \alpha(p)\, L_{||}(p) \tag{1.7}$$

which implies

$$\sigma \leqslant \nu'_{||} - \nu'_{\perp} \tag{1.8}$$

If Section 4 we will introduce yet another definition of the parallel correlation length in the supercritical case: $L^{\varepsilon}_{||}(p) =$ the smallest length for which the renormalized bond construction of ref. 3 works. This definition is analogous to the definition in terms of sponge crossings for ordinary percolation.[9] We would like to show that this definition is (up to constants) the same as $L_{||}(p)$, but all we can show is

$$L^{\varepsilon}_{||}(p) \geqslant 2 \log 2\, L_{||}(p) \tag{1.9}$$

In Section 5 we use this definition to prove that if $L = L^{\varepsilon}_{||}(p)$,

$$P_p(\Omega_{\infty}) \geqslant (1/2)\, P_p(\xi^0_L \neq \varnothing) \tag{1.10}$$

The last inequality says that percolation is almost the same as surviving up to the correlation length. The proof of Lemma 4.1 in ref. 11 also works in this case to show (1.10) when we take $L = 4dL_{||}(p)|\log L_{||}(p)|$. In fact, the result in ref. 11 is more general in the sense that it is true for any finite dimension d. In terms of critical exponents, (1.10) says

$$\beta \leqslant \nu_{||}^{\varepsilon}/\delta_r \tag{1.11}$$

An extension of the proof of (1.10) gives

$$P_p(x \in \xi_{2L}^0) \asymp [P_p(\Omega_\infty)]^2 \qquad \text{for} \quad |x| \leqslant 1.5\alpha(p)L \tag{1.12}$$

where $\asymp$ means the ratio of these quantities is bounded above and away from 0 by constants independent of p, and again we have written L for $L_{||}^{\varepsilon}(p)$. If we define η' by

$$E_p|\xi_{2L}^0| \approx [L]^{-\eta'} \approx (p - p_c)^{\nu_{||}'\eta'}$$

(trusting here that quantities at the correlation length are, up to constants, the same as at p_c), then (1.12) leads to

$$\nu_{||}'\eta' \geqslant \nu_{||}' - \sigma - 2\beta \tag{1.13}$$

The last result is one-half of

$$\nu_{||}'\eta' = \nu_{||}' - \sigma - 2\beta \tag{1.14}$$

a relationship which follows from (1.1), (1.2), and (1.4).

Finally, we have

$$\gamma \leqslant (\eta + 1)\nu_{||} \tag{1.15}$$

If one remembers $\eta = 1 - \eta'$, then one will recognize this as Fisher's inequality.[6] If one defines the connectivity radius as the random variable $R_{||}$ whose distribution is given by

$$P_p(R_{||} = n) = \frac{\sum_x P_p((0,0) \to (x, n))}{E_p|C_0|}$$

and defines exponents ν_k by

$$\{E_p(R_{||})^k\}^{1/k} \approx (p_c - p)^{-\nu_k}$$

then one can use ideas of ref. 10 to show

$$\lim_{k \to \infty} \nu_k = \nu_{||}$$

and

$$\gamma \leqslant (\eta + 1)\nu_{||}$$

No new ideas are needed, so the proof is omitted.

2. SCALING INEQUALITY FOR THE SUBCRITICAL PROCESS

In this section we will prove that, for $p < p_c$ and $L_\perp(p) \geqslant 1$,

$$E_p|C_0| \leqslant 10 L_\perp(p)\, L_{||}(p) \tag{2.1}$$

This relationship is natural if we notice that $L_{||}(p)$ and $L_\perp(p)$ give the height and width of a typical cluster, while $|C_0|$ gives its volume. To see where the missing 2β in the associated exponent inequality (1.6) should come from, look at (1.12).

Proof. From the definitions of the correlation lengths we have that

$$P_p((0, 0) \to (x, n)) \leqslant \min\{\exp[-n/L_{||}(p)], \exp[-|x|/L_\perp(p)]\}$$

Set $c = L_\perp(p)/L_{||}(p)$. Let $A = \{(x, n)\colon |x| \leqslant cn\}$, where $\exp[-n/L_{||}(p)] \leqslant \exp[-|w|/L_\perp(p)]$ and $B = A^c \cap \{(x, n)\colon n \geqslant 0\}$, where the opposite inequality holds. Then

$$\begin{aligned}
E_p|C_0| &= \sum_n \sum_x P_p((0, 0) \to (x, n)) \\
&= \sum_A P_p((0, 0) \to (x, n)) + \sum_B P_p((0, 0) \to (x, n)) \\
&\leqslant \sum_n (2[cn] + 1) \exp\{-n/L_{||}(p)\} + \sum_x [c^{-1}|x|] \exp\{-|x|/L_\perp(p)\}
\end{aligned}$$

where $[a]$ is the largest integer $\leqslant a$. Using the trivial inequality

$$\sum_{k=1}^{\infty} k e^{-ak} \leqslant \int_0^\infty (x+1) e^{-ax}\, dx = a^{-2} + a^{-1}$$

we see that the expression above is

$$\begin{aligned}
&\leqslant 1 + 2c(\{L_{||}(p)\}^2 + L_{||}(p)) + L_{||}(p) + 2c^{-1}(\{L_\perp(p)\}^2 + L_\perp(p)) \\
&\leqslant 10 L_{||}(p)\, L_\perp(p) \qquad \text{when} \quad L_\perp(p), L||(p) \geqslant 1 \quad \blacksquare
\end{aligned}$$

Corollary. The following relation holds:

$$\gamma \leqslant \nu_{||} + \nu_\perp \tag{2.2}$$

3. ONE SCALING INEQUALITY FOR THE EDGE SPEED

We begin with some definitions. Set

$$l_n^0 = \inf \xi_n^0, \qquad \inf \varnothing = +\infty$$

$$r_n^0 = \sup \xi_n^0, \qquad \sup \varnothing = -\infty$$

$$G_n = \{l_n^0 \geqslant -(1+\delta)\,\alpha(p)n,\ r_n^0 \leqslant (1+\delta)\,\alpha(p)n\}$$

where $\delta > 0$. Let

$$\xi_n^{(A,m)} = \{y\colon \text{ for some } x \in A,\ (x, m) \to (y, n)\}$$

$$H_n = \{\tau^{[-(1+\delta)\alpha(p)n,(1+\delta)\alpha(p)n]} < \infty\}$$

From the definitions above it should be clear that

$$P(n < \tau^0 < \infty) \geqslant P(\xi_n^0 \neq \varnothing,\ G_n,\ \xi_m^{(\xi_n^0, n))} \text{ dies out})$$

$$\geqslant P(\xi_n^0 \neq \varnothing,\ G_n)\ P(H_n)$$

In ref. 2 it was shown that

$$P(G_n \mid \xi_n^0 \neq \varnothing) \to 1 \qquad \text{as} \quad n \to \infty$$

so we have

$$P(\xi_n^0 \neq \varnothing,\ G_n) \to P(\Omega_\infty) > 0$$

On the other hand,

$$(1/n) \log P(H_n) \to -\alpha(p)(1+\delta)/L_\perp^\tau(p)$$

and since δ is arbitrary, if follows that

$$-1/L_{||}(p) \geqslant -\alpha(p)/L_\perp^\tau(p)$$

or, rearranging,

$$L_\perp^\tau(p) \leqslant \alpha(p)\ L_{||}(p) \tag{3.1}$$

In terms of critical exponents, we have obtained

$$\sigma \leqslant \nu_{||}' - \nu_\perp' \tag{3.2}$$

4. RENORMALIZED BOND CONSTRUCTION

The first thing to do is to describe a construction due to Durrett and Griffeath.[3] We follow the version in ref. 2. Let G be the graph with vertices in $\mathscr{L}_0 = \{(m, n): m+n \text{ is even}, n \geqslant 0\}$ and with oriented bonds connecting each $(m, n) \in \mathscr{L}_0$ to $(m+1, n+1)$ and to $(m-1, n+1)$. Consider the "renormalized lattice" $\mathscr{L}_0$ to be mapped into the upper half plane $\mathbb{R} \times [0, \infty)$ by $\phi(m, n) = (aLm, Ln)$, where a is a special constant and L is a large number, both to be chosen below.

To each $z \in \mathscr{L}_0$ we associate a random variable $\eta(z)$ such that $\eta(z) = 1$ if a certain "good event" happens near $\phi(z)$ in our original percolation process and $\eta(z) = 0$ otherwise. This procedure generates a 1-dependent, oriented site percolation process with $\eta(z) = 1$, meaning that the site z is open, and $\eta(z) = 0$, meaning that the site z is closed. We call this new process the rescaled process.

The choices of the constants and of the "good events" are made in a such a way that:

(i) The random variables $\eta(z)$, $z \in \mathscr{L}_0$, are 1-dependent, i.e., if we let $\|(m, n)\|_{\mathscr{L}} = (|m| + |n|)/2$ and $z_1, ..., z_m$ are points with $\|z_i - z_j\|_{\mathscr{L}} > 1$ for $i \neq j$, then $\eta(z_1), ..., \eta(z_m)$ are independent.

(ii) If L is large, then the probability that $\eta(x) = 1$ is close to 1.

(iii) If percolation occurs in the η-process starting from the origin, then the same thing happens in the original process starting from some point near the origin.

To introduce the "good event," let A be the parallelogram with vertices

$$u_0 = (-0.1\alpha L, 0), \qquad v_0 = (0.1\alpha L, 0)$$

$$u_1 = ((1-0.05)\,\alpha L, (1+0.05)L), \qquad v_1 = ((1+0.15)\,\alpha L, (1+0.05)L)$$

We associate the sites in $\mathscr{L}_0$ with translations of A in the original percolation structure: If we let $v_{m,n} = ((\alpha - 4\delta)m, n)$ for $(m, n) \in \mathscr{L}_0$, then we define the translations of A by

$$A_{m,n} = (v_{m,n} + (-4\delta\alpha, 0)) \cdot L + A$$

$$B_{m,n} = (v_{m,n} + (4\delta\alpha, 0)) \cdot L - A$$

where $x - A = \{x - y: y \in A\}$. For a picture see Fig. 1.

The "good event" for the site (m, n) happens if there are open paths from top to bottom lying entirely in $A_{m,n}$ and in $B_{m,n}$. If we denote the good event by $SC(L)$ (for sponge crossing), then it is known (see ref. 2, Section 9) that, for $p > p_c$,

$$P_p(SC(L)) \to 1 \qquad \text{as} \quad L \to \infty \tag{4.1}$$

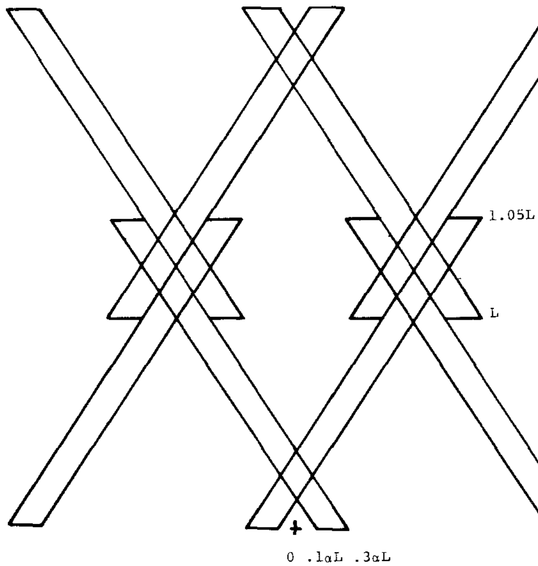

Fig. 1.

Again by ref. 2, now in Section 10, if

$$P_p(SC(L)) \geqslant 1 - 6^{-36} \tag{4.2}$$

then the probability that the η-system percolates is greater than 1/2.

Let $\varepsilon_0 = 6^{-36}$ and define

$$L_{||}^{\varepsilon}(p) = \inf\{n\colon\ P_p(SC(n)) \geqslant 1 - \varepsilon_0\}$$

Let

$$\xi_m = \{x\colon\ (x, m) \in \mathscr{L}_0 \text{ and there is an } y \leqslant 0 \text{ such that } (y, 0) \in \mathscr{L}_0 \text{ and } (y, 0) \to (x, m)\}$$

where $(y, 0) \to (x, m)$ means there is an open path from $(y, 0)$ to (x, m) on the renormalized lattice. Finally, let

$$\bar{s}_m = \sup \xi_m$$

Relation (2) in Section 11 of ref. 2 applied to our process $\{\eta\}$ gives us that, if $p > p_c$, then

$$P(\bar{s}_k \leqslant 0) \leqslant (1/2)^{k-1} \tag{4.3}$$

Write L for $L_{||}^{\varepsilon}(p)$ and consider the mapping $\bar{R}_{\varepsilon,k}$ from $\mathscr{L}_0$ into $\mathbb{R}$ given by

$$\bar{R}_{\varepsilon,k} = \bar{s}_k \cdot \alpha \cdot L$$

If $\bar{r}_n$ is the right edge of the original process starting with the configuration $\{0, -2, -4,...\}$, then it follows from the construction that

$$\bar{R}_{\varepsilon,k}(\omega) \leqslant \bar{r}_{(k+1)L}(\omega) \qquad \text{for every } \omega \tag{4.4}$$

In ref. 4 it was shown that, for any $p \in [0, 1]$,

$$\lim_{n \to \infty} \left\{ -\frac{1}{n} \log P_p(\bar{r}_n \leqslant 0) \right\} = [2L_{||}(p)]^{-1} \tag{4.5}$$

With these results in hard, we can get that, for $p > p_c$,

$$L_{||}^{\varepsilon}(p) \geqslant 2 \log 2\, L_{||}(p) \tag{4.6}$$

Proof. By using (4.3) and (4.4), it is not difficult to see that

$$P_p(\bar{r}_{(k+1)L} \leqslant 0) \leqslant P_p(\bar{R}_{\varepsilon,k} \leqslant 0) \leqslant P_\varepsilon(\bar{s}_k \leqslant 0) \leqslant (1/2)^{k-1}$$

so that

$$-\frac{1}{(k+1)L} \log P_p(\bar{r}_{(k+1)L} \leqslant 0) \geqslant \frac{\log 2}{L} \frac{k-1}{k+1}$$

By letting $k \uparrow \infty$ and recalling (4.5), we get that

$$[2L_{||}(p)]^{-1} \geqslant \frac{\log 2}{L_{||}^{\varepsilon}(p)} \quad \blacksquare$$

5. PROOFS OF (1.7) AND (1.10)

Throughout this section we will write L for $L_{||}^{\varepsilon}(p)$.
If $p > p_c$, then

$$(1/2)\, P_p(\xi_L^0 \neq \varnothing) \leqslant P_p(\Omega_\infty) \leqslant P_p(\xi_L^0 \neq \varnothing) \tag{5.1}$$

Proof. The right-hand inequality is obvious. To prove the other one, we note two things: First, that the event $W = \{$the rescaled process percolates$\}$ has probability $\geqslant 1/2$ by (4.2). Second, in order to have a path from zero to infinity on the event W it suffices that the process starting with configuration $\{0\}$ survives until time L, since if ξ_t^0 survives until time

L, then it crosses out at least one of the paths involved in the event $A_{00} \cap B_{00}$ (see Fig. 2). Using Harris–FKG inequality,[8] now gives

$$P_p(\Omega_\infty) \geqslant P_p(\{\xi_L^0 \neq \varnothing\} \cap W) \geqslant P_p(\xi_L^0 \neq \varnothing)(1/2) \tag{5.1}$$

and the proof is complete. ∎

Corollary. The following relation holds:

$$\beta \leqslant \nu'_{\|}/\delta_r \tag{5.2}$$

Proof. We have that

$$P_p(\xi_L^0 \neq \varnothing) \geqslant P_{\mathrm{cr}}(\xi_L^0 \neq \varnothing) \approx (L)^{-1/\delta_r} \approx (p - p_c)^{\nu'_{\|}/\delta_r}$$

and from the previous proposition it follows that

$$P_p(\xi_L^0 \neq \varnothing) \asymp P_p(\Omega_\infty) \approx (p - p_c)^\beta \quad ∎$$

For $p > p_c$ and $|x| \leqslant 1.5\alpha L$ there are constants c and $C \in (0, \infty)$ so that

$$c(P_p(\Omega_\infty))^2 \leqslant P_p(x \in \xi_{2L}^0) \leqslant C(P_p(\Omega_\infty))^2 \tag{5.3}$$

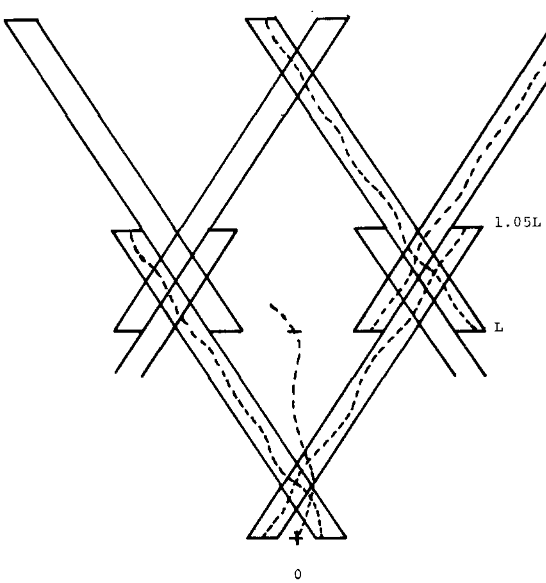

Fig. 2.

Proof. Note that if for convenience we choose t to be even, then

$$\begin{aligned}P_p(x \in \xi_t^0) &= P_p(\xi_{t/2}^0 \neq \varnothing, \xi_t^{(\xi_{t/2}^0, t/2)}(x) = 1)\\ &= P_p(\xi_{t/2}^0 \neq \varnothing)\, P_p(\xi_t^{(t/2, \xi_{t/2}^0)}(x) = 1 \mid \xi_{t/2}^0 \neq \varnothing)\\ &\leqslant P_p(\xi_{t/2}^0 \neq \varnothing)\, P_p(\xi_{t/2}^Z(x) = 1)\end{aligned}$$

Since the probability of a path from $Zx\{0\}$ to $(x, t/2)$ is the same as that of a path from $(x, 0)$ to $Zx\{t\}$ (see ref. 2, Section 8),

$$P_p(\xi_{t/2}^Z(x) = 1) = P_p(\xi_{t/2}^{\{x\}} \neq \varnothing) = P_p(\xi_{t/2}^0 \neq \varnothing)$$

Combining the last two equations gives

$$P_p(x \in \xi_t^0) \leqslant \{P_p(\xi_{t/2}^0 \neq \varnothing)\}^2$$

By choosing $t = 2L$ and recalling (5.1), one can get

$$P_p(x \in \xi_{2L}^0) \leqslant \{P_p(\Omega_\infty)/2\}^2$$

This proves the right-hand inequality. For the other half, let

$F = \{$the sites $(0, 0)$, $(1, 1)$, $(-1, 1)$ are open in the rescaled process$\}$

$G = \{\xi_L^0 \neq \varnothing\}$

$H = \{x \in \xi_{2L}^{(Z, L)}\}$

We claim that if $|x| \leqslant 1.5\alpha L$, then on $F \cap G \cap H$, $x \in \xi_{2L}^0$. To see this, look at Fig. 3 and notice that (i) when F occurs, there are paths inside each one of the six parallelograms; (ii) when G occurs, the origin is connected to at least one of the open paths in A_{00} and B_{00}; (iii) when H occurs, one can get from one of parallelograms to the point $(x, 2L)$.

Combining the observations above, we obtain

$$P_p(x \in \xi_{2L}^0) \geqslant P_p(F \cap G \cap H) \geqslant P_p(F)\, P_p(G)\, P_p(H)$$

by the Harris–FKG inequality, since all three events are increasing. Since each rescaled site is closed with probability ε_0,

$$P_p(F) \geqslant 1 - 3\varepsilon_0$$

By definition,

$$P_p(G) = P_p(\xi_L^0 \neq \varnothing)$$

Finally, as we observed in the first half of the proof,

$$P_p(H) = P_p(x \in \xi_{2L}^{(Z, L)}) = P_p(\xi_L^x \neq \varnothing) = P_p(\xi_L^0 \neq \varnothing)$$

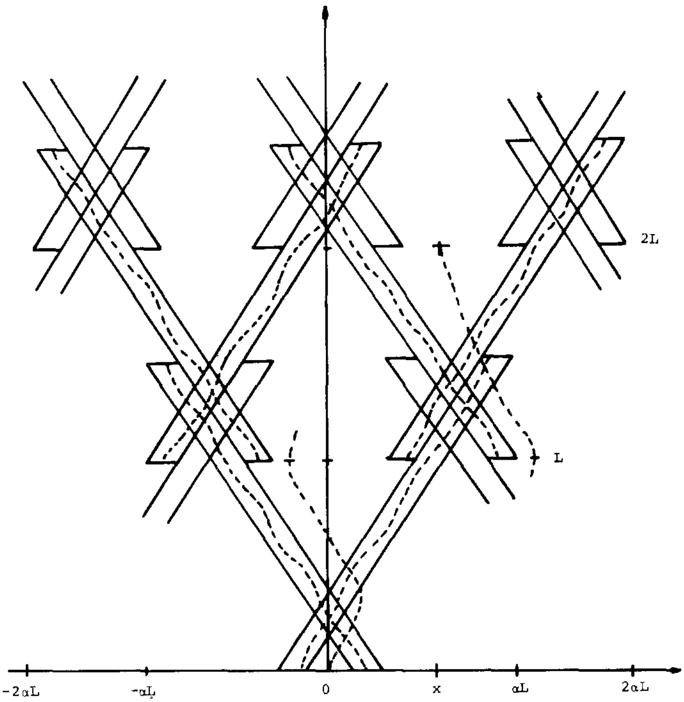

Fig. 3.

Putting the last four observations together, it follows that

$$P_p(x \in \xi_{2L}^0) \geqslant \{P_p(\xi_L^0 \neq \varnothing)\}^2 (1 - 3\varepsilon_0)$$

and the desired result folows from (5.1). ▮

Corollary. The following relation holds:

$$v'_{\|}\eta' \geqslant v'_{\|} - \sigma - 2\beta \tag{5.4}$$

Proof. Recall that by definition

$$E_p |\xi_{2L}^0| \approx (L)^{\eta'} \approx (p - p_c)^{-v'_{\|}\eta'}$$

The last proposition implies that

$$E_p |\xi_{2L}^0| \geqslant \sum_{|x| \leqslant 1.5\alpha L} P_p(x \in \xi_{2L}^0) \geqslant 3\alpha L \cdot c\{P_p(\Omega_\infty)\}^2$$

which implies the result stated in the corollary. ▮

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